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ORIGINAL PAPER

The noisy Pais–Uhlenbeck oscillator

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Abstract
In this paper, we include simultaneously additive and multiplicative noise to the Pais–Uhlenbeck oscillator (PUO). We construct an integral of motion of the PUO with a time-dependent coefficient. Viewing the PUO as two coupled harmonic oscillators, we add noise to the corresponding frequencies. The systems are solved with the fourth-order stochastic Runge–Kutta method. Some graphics of the solutions and integrals of motion are presented, and the average deviations are calculated in order to quantify the noise influence.

Keywords Pais–Uhlenbeck oscillator · Additive noise · Multiplicative noise · Integral of motion · Runge–Kutta method

Mathematics Subject Classification 34F05 · 60H10 · 93E03

1 Introduction

One of the best-known models with high-order derivatives is the Pais–Uhlenbeck oscillator (PUO) [1]. This model was originally proposed in order to solve divergence problems in quantum field theory. Several studies on this model have been performed in the literature, like some Hamiltonian formulations and their canonical quantizations [2], some supersymmetric extensions [3,4], investigations on ghosts and unitarity violations [5] or some stability analyzes [6]. In particular, Bolonek and Kosinski [7]

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carried out an exhaustive study on the Hamiltonian structure of PUO using its integrals of motion.

The PUO is usually written in the following form:

$$\frac{d^4x}{dt^4} + c\frac{d^2x}{dt^2} + ex = f(t). \tag{1.1}$$

Here, c and e are real constants. Note that we have included the forced term $f(t)$. This oscillator can be interpreted as two coupled harmonic oscillators with frequencies ω_1 and ω_2, where $c = \omega_1^2 + \omega_2^2$ and $e = \omega_1^2\omega_2^2$ (see [2,7]). In chemistry, the inclusion of noise has been an efficient tool to model environmental effects in the chemical kinetics [8], chirality [9] or electron transfer reactions [10]. In particular, noisy coupled oscillator systems have been used to model solid state dynamics at low temperatures [11] or photosynthetic systems [12]. On the other hand, stochastic differential equations can be useful to model systems in physics, economy, biology, etc. [13,14].

The aim of this work is to consider possible stochastic perturbations in the form of white noise in the PUO. This idea has been widely discussed by Gitterman in [15] for the harmonic and some nonlinear oscillators, and also in Ermakov–Lewis systems and their invariants [16–18]. In addition, there are works on numerical simulations of stochastic linear oscillators [19]. In particular, we are interested in quantifying the influence on the solutions of PUO under little perturbations. As it is well known, the noise can be included in additive form (which can be interpreted as an external perturbation to the system) and multiplicatively (that is, viewed as internal noise). Of course, these noises can be implemented simultaneously. In this sense, the additive noise can be included directly in (1.1) in the usual way. However, there are at least two ways to include the multiplicative noise: the simplest case where it is included on the dependent variable $x(t)$, or as a perturbation on the primary frequencies ω_1 and ω_2. Also, it is possible to investigate noise effects on an integral of motion of the PUO.

This paper is organized as follows. In Sect. 2, we include additive and multiplicative noise to the PUO in the usual form. In Sect. 3, we construct an integral of motion for the PUO but taking the coefficient e as time dependent, and also include additive and multiplicative noise on it. In Sect. 4, we consider the inclusion of additive and multiplicative noise, but the latter is added as a perturbation on the primary frequencies of the oscillator. In Sect. 5, the results are discussed and, finally, a brief conclusion is presented.

2 Noise in the PUO

To include simultaneously both additive and multiplicative noises in the equation (1.1), it is convenient to rewrite the mathematical model in the form

$$dX_t = a(t, X_t)dt + A(t, X_t)dA_t + M(t, X_t)dM_t, \tag{2.1}$$

where A_t and M_t are the normalized stochastic variables, and the vectors dX_t, $a(t, X_t)$, $A(t, X_t)$ and $M(t, X_t)$ are

$$\begin{cases} dX_t = (dx, d\dot{x}, d\ddot{x}, d\dddot{x})^\top, \\ a(t, X_t) = (\dot{x}, \ddot{x}, \dddot{x}, -c\ddot{x} - ex + f(t))^\top, \\ A(t, X_t) = (0, 0, 0, \alpha_A)^\top, \\ M(t, X_t) = (0, 0, 0, \alpha_M x^n)^\top. \end{cases} \tag{2.2}$$

The point means temporal derivative, α_A is the additive noise intensity, α_M is the multiplicative noise intensity and the power n is equal to 1. We can use a general function of x for multiplicative noise, but we preferred to use the simplest possible form.

To obtain the numerical results, we let $c = 5$ and $e = 4$, and use the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$, $\ddot{x}(0) = 1$ and $\dddot{x}(0) = 0$. We will study the homogeneous PUO with $f(t) = 0$ and the forced PUO with $f(t) = \sin^2(3t)$. For the intensity of additive noise, we use $\alpha_A = 0.02, 0.1$, that approximately correspond to 1% and 5% of the amplitude of the solution, respectively. For the multiplicative noise, we employ $\alpha_M = 0.04, 0.2$, which correspond respectively to 1% and 5% of the parameter e. We solve the equation (2.1) using the fourth-order Runge–Kutta method. This method has been preferred in light that it yields better results when compared with others approaches [20]. We generated the normalized stochastic variable using the Box-Müller algorithm [21]. The results for simultaneous additive–multiplicative 1% and 5% noise for homogeneous PUO and forced PUO are shown in Fig. 1.

To measure quantitatively the noise influence, we get the corresponding average deviation S of the noisy solutions with respect to the noiseless solution.[1] The results are shown in Table 1. Obviously, we can see more noise influence as the percentage increases. However, note that the additive noise affects similarly the homogeneous and the forced PUO, but the multiplicative noise perturbs more the solutions than the additive one in both cases (see for example rows 2 and 6 of Table 1).

3 Noise in an integral of motion

Note that in the previous section, the multiplicative noise can be viewed as a time-dependent coefficient of the PUO (1.1), which means that it can be written as

$$\frac{d^4x}{dt^4} + c\frac{d^2x}{dt^2} + e(t)x = f(t), \tag{3.1}$$

where the additive noise can be added to the forced term $f(t)$ in similar form to the previous section. On the other hand, we can find two integrals of motion (also called

[1] We calculate S as

$$S = \sqrt{\frac{\sum_{i=1}^{N} \left(x_{i,nonoise} - x_{i,noise}\right)^2}{N}} \tag{2.3}$$

where N is the number of steps in the Runge–Kutta method.

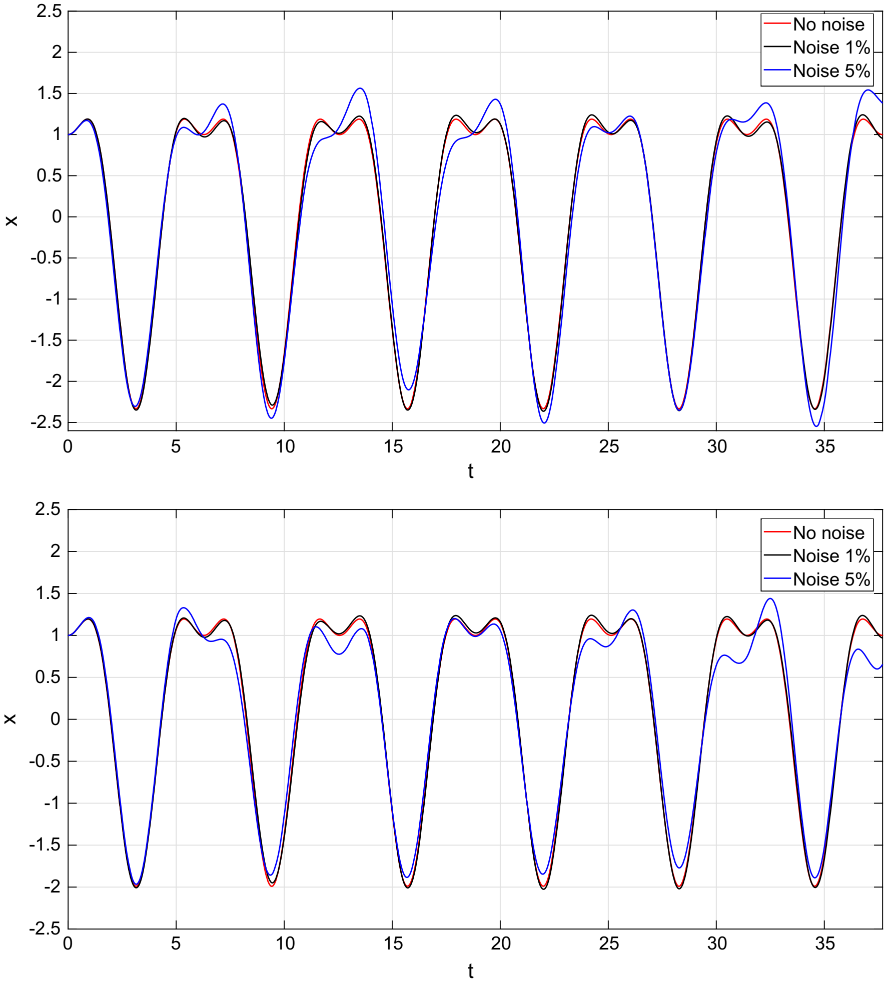

Fig. 1 Solution of homogeneous (up) and forced (down) PUO with simultaneous additive–multiplicative noise. Red color corresponds to the solution without noise, black color corresponds to the solution with noise intensity $\alpha_A = 0.02$, $\alpha_M = 0.04$ (1%) and blue color corresponds to the solution with noise intensity $\alpha_A = 0.1$, $\alpha_M = 0.2$ (5%) (Color figure online)

dynamical invariants) for (1.1) in [7]. However, the situation is different if we look for an integral of motion for the non-autonomous PUO (3.1). In that case, the problem is similar to the well-known Ermakov–Lewis invariant, which is associated to the integral of motion of an oscillator where the frequency dependents on time [22,23]. In this section, we will obtain an integral of motion for (3.1), following the dynamic algebraic formalism proposed by Korsch [24]. Afterwards, we will show the noise effects on the integral of motion, paraphrasing the study of the noisy Ermakov–Lewis invariants [16–18].

Table 1 Average deviations of the homogeneous and the forced PUO with all possible combinations of 1% and 5% simultaneous additive–multiplicative noise

Additive noise percent (%)	Multiplicative noise percent (%)	Average deviation S
Homogeneous PUO		
1	0	0.01246
5	0	0.07544
0	1	0.04403
1	1	0.03205
5	1	0.08023
0	5	0.16178
1	5	0.17703
5	5	0.17914
Forced PUO		
1	0	0.01326
5	0	0.06396
0	1	0.02774
1	1	0.03124
5	1	0.07977
0	5	0.11285
1	5	0.20076
5	5	0.17168

The first step is to obtain a Lagrangian associated to the Eq. (3.1). To achieve this goal, we write the Euler–Lagrange (EL) equation for a high-order derivative Lagrangian of the type $L = L(x, \dot{x}, \ddot{x}, t)$ [25]:

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0. \tag{3.2}$$

Expanding the terms of the EL Eq. (3.2) with the Lagrangian $L = L(x, \dot{x}, \ddot{x}, t)$ and comparing with the terms in the equation of motion (3.1), we obtain the Lagrangian

$$L(x, \dot{x}, \ddot{x}, t) = \frac{1}{2}\ddot{x}^2 - \frac{c}{2}\dot{x}^2 + \frac{1}{2}e(t)x^2 - f(t)x. \tag{3.3}$$

Now, we will use the Ostrogradsky's formalism, which has been designed for Lagrangians and Hamiltonians containing derivatives of higher order [26]. First, one should get the new coordinates and momenta. These are defined generically as follows:

$$q_j = \frac{d^{j-1}}{dt^{j-1}}x, \tag{3.4}$$

$$p_j = \sum_{r=1}^{n}(-1)^{r-j}\frac{d^{r-j}}{dt^{r-j}}\frac{\partial L}{\partial x^{(r)}}. \tag{3.5}$$

In the particular case of (3.3), they assume the form $(q_1, q_2, p_1, p_2) = (x, \dot{x}, -c\dot{x} - \dddot{x}, \ddot{x})$. The Hamiltonian is obtained then by means of the usual Legendre transformation

$$H = \frac{1}{2}p_2^2 + p_1 q_2 - \frac{1}{2}e(t)q_1^2 + \frac{c}{2}q_2^2 + f(t)q_1. \tag{3.6}$$

Using the conventional Hamilton equations, it is easy to prove that they provide the equation of motion (3.1).

Following the dynamic algebraic formalism [24], we propose the following basis for a representation of the Lie algebra using the terms of the Hamiltonian (3.6):

$$\begin{cases} L_1 = I, & L_6 = q_1q_2, & L_{12} = q_1^2, \\ L_2 = q_1, & L_7 = q_1p_1, & L_{13} = q_2^2, \\ L_3 = q_2, & L_8 = q_1p_2, & L_{14} = p_1^2, \\ L_4 = p_1, & L_9 = q_2p_1, & L_{15} = p_2^2, \\ L_5 = p_2, & L_{10} = q_2p_2, & L_{11} = p_1p_2. \end{cases} \tag{3.7}$$

Thus the Hamiltonian (3.6) written in terms of the basis (3.7) will be

$$H = \frac{1}{2}L_{15} + L_9 - \frac{1}{2}e(t)L_{12} + \frac{c}{2}L_{13} + f(t)L_2, \tag{3.8}$$

where we have eliminated the term proportional to L_1 since the identity commutes with all the elements of the basis. The commutator is the Poisson bracket, which is defined in the following way [27]:

$$\{f, g\} = \frac{\partial f}{\partial p_1}\frac{\partial g}{\partial q_1} - \frac{\partial f}{\partial q_1}\frac{\partial g}{\partial p_1} + \frac{\partial f}{\partial p_2}\frac{\partial g}{\partial q_2} - \frac{\partial f}{\partial q_2}\frac{\partial g}{\partial p_2}. \tag{3.9}$$

Therefore, one obtains the nontrivial commutators

$$\begin{cases} \{L_2, L_4\} = -L_1, & \{L_4, L_7\} = L_4, & \{L_8, L_{11}\} = -L_{15}, & \{L_3, L_5\} = -L_1, \\ \{L_5, L_9\} = L_4, & \{L_9, L_{12}\} = 2L_6, & \{L_4, L_6\} = L_3, & \{L_6, L_8\} = -L_{12}, \\ \{L_{10}, L_{15}\} = -2L_{15}, & \{L_5, L_6\} = L_2, & \{L_6, L_{15}\} = -2L_8, & \{L_2, L_{11}\} = -L_5, \\ \{L_6, L_7\} = -L_6, & \{L_7, L_9\} = -L_9, & \{L_3, L_{11}\} = -L_4, & \{L_6, L_{14}\} = -2L_9, \\ \{L_8, L_{10}\} = L_8, & \{L_4, L_{12}\} = 2L_2, & \{L_7, L_8\} = L_8, & \{L_9, L_{11}\} = -L_{14}, \\ \{L_5, L_{13}\} = 2L_3, & \{L_8, L_9\} = L_7 - L_{10}, & \{L_{10}, L_{13}\} = 2L_{13}, & \{L_6, L_{10}\} = -L_6, \\ \{L_9, L_{10}\} = -L_9, & \{L_{11}, L_{13}\} = 2L_9, & \{L_7, L_{12}\} = 2L_{12}, & \{L_{10}, L_{11}\} = -L_{11}, \\ \{L_2, L_9\} = -L_3, & \{L_8, L_{13}\} = 2L_6, & \{L_{11}, L_{12}\} = 2L_8, & \{L_3, L_{10}\} = -L_3, \\ \{L_9, L_{15}\} = -2L_{11}, & \{L_{12}, L_{14}\} = -4L_7, & \{L_4, L_8\} = L_5, & \{L_2, L_{14}\} = -2L_4, \\ \{L_{13}, L_{15}\} = -4L_{10}, & \{L_5, L_{10}\} = L_5, & \{L_3, L_{15}\} = -2L_5, & \{L_2, L_7\} = -L_2, \\ \{L_6, L_9\} = -L_{13}, & \{L_6, L_{11}\} = -L_7 - L_{10}, & \{L_3, L_8\} = -L_2, & \{L_7, L_{11}\} = -L_{11}, \\ & \{L_7, L_{14}\} = -2L_{14}, & \{L_8, L_{14}\} = -2L_{11}. & \end{cases} \tag{3.10}$$

The integral of motion will be a linear combination of the elements of the basis: $I = \sum \lambda_i(t)L_i$. This linear combination will be an integral of motion if it satisfies the well-known equation $\{H, I\} + \frac{\partial I}{\partial t} = 0$ [27]. Using substitutions and the commutators (3.10), we obtain the following system of equations for the coefficients λ:

$$
\begin{cases}
\dot{\lambda}_1(t) = f(t)\lambda_4(t), & \dot{\lambda}_2(t) = f(t)\lambda_7(t) - e(t)\lambda_4(t), \\
\dot{\lambda}_3(t) = c\lambda_5(t) + f(t)\lambda_9(t) - \lambda_2(t), & \dot{\lambda}_4(t) = \lambda_5(t) + 2f(t)\lambda_{14}(t), \\
\dot{\lambda}_5(t) = f(t)\lambda_{11}(t) - \lambda_3(t), & \dot{\lambda}_6(t) = c\lambda_8(t) - 2\lambda_{12}(t) - e(t)\lambda_9(t), \\
\dot{\lambda}_7(t) = \lambda_8(t) - 2e(t)\lambda_{14}(t), & \dot{\lambda}_8(t) = -\lambda_6(t) - e(t)\lambda_{11}(t), \\
\dot{\lambda}_9(t) = \lambda_{10}(t) + c\lambda_{11}(t) - \lambda_7(t), & \dot{\lambda}_{10}(t) = 2c\lambda_{15}(t) - 2\lambda_{13}(t) - \lambda_8(t), \\
\dot{\lambda}_{11}(t) = 2\lambda_{15}(t) - \lambda_9(t), & \dot{\lambda}_{12}(t) = -e(t)\lambda_7, \\
\dot{\lambda}_{13}(t) = c\lambda_{10}(t) - \lambda_6(t), & \dot{\lambda}_{14}(t) = \lambda_{11}(t), \\
\dot{\lambda}_{15}(t) = -\lambda_{10}(t).
\end{cases}
\tag{3.11}
$$

The solution of the system of equations (3.11) is

$$
\begin{cases}
\lambda_1(t) = \int f(t)\psi(t)dt, \\
\lambda_2(t) = -4f(t)\ddot{\rho}(t) - 5\dot{f}(t)\dot{\rho}(t) - 2\ddot{f}(t)\rho(t) - 2cf(t)\rho(t) \\
\qquad -2f(t)\phi(t) + \dddot{\psi}(t) + c\dot{\psi}(t), \\
\lambda_3(t) = 3f(t)\dot{\rho}(t) + 2\dot{f}(t)\rho(t) - \ddot{\psi}(t), \\
\lambda_4(t) = \psi(t), \\
\lambda_5(t) = -2f(t)\rho(t) + \dot{\psi}(t), \\
\lambda_6(t) = -\rho^{(5)}(t) - c\dddot{\rho}(t) - 3e(t)\dot{\rho}(t) - 2\dot{e}(t)\rho(t) - 3\dddot{\phi}(t), \\
\lambda_7(t) = \dddot{\rho}(t) + c\dot{\rho}(t) + 3\dot{\phi}(t), \\
\lambda_8(t) = \rho^{(4)}(t) + c\ddot{\rho}(t) + 2e(t)\rho(t) + 3\ddot{\phi}(t), \\
\lambda_9(t) = -\ddot{\rho}(t) - 2\phi(t), \\
\lambda_{10}(t) = \dot{\phi}(t), \\
\lambda_{11}(t) = \dot{\rho}(t), \\
\lambda_{12}(t) = \frac{1}{2}\rho^{(6)}(t) + c\rho^{(4)}(t) + 2e(t)\ddot{\rho}(t) + \frac{c^2}{2}\ddot{\rho}(t) + \frac{5}{2}\dot{e}(t)\dot{\rho}(t) + \ddot{e}(t)\rho(t) \\
\qquad +ce(t)\rho(t) + \frac{3}{2}\phi^{(4)}(t) + \frac{3c}{2}\ddot{\phi}(t) + e(t)\phi(t), \\
\lambda_{13}(t) = -\frac{1}{2}\rho^{(4)}(t) - \frac{c}{2}\ddot{\rho}(t) - e(t)\rho(t) - 2\ddot{\phi}(t) - c\phi(t), \\
\lambda_{14}(t) = \rho(t), \\
\lambda_{15}(t) = -\phi(t).
\end{cases}
\tag{3.12}
$$

Note that the function $\psi(t)$ must satisfy the homogeneous equation of the PUO, namely,

$$\psi^{(4)}(t) + c\ddot{\psi}(t) + e(t)\psi(t) = 0. \tag{3.13}$$

Meanwhile, the function $\phi(t)$ satisfies the following set of auxiliary equations:

$$5f(t)\dot{\phi}(t) + 2\dot{f}(t)\phi(t) = 0, \tag{3.14}$$

$$\frac{3}{2}\phi^{(5)}(t) + \frac{3c}{2}\dddot{\phi}(t) + 4e(t)\dot{\phi}(t) + \dot{e}(t)\phi(t) = 0, \tag{3.15}$$

$$5\dddot{\phi}(t) + 2c\dot{\phi}(t) = 0. \tag{3.16}$$

On the other hand, the function $\rho(t)$ must satisfy simultaneously the following equations

$$5f(t)\dddot{\rho}(t) - 9\dot{f}(t)\ddot{\rho}(t) - \left[7\ddot{f}(t) + 3cf(t)\right]\dot{\rho}(t) - 2\left[\dddot{f}(t) + c\dot{f}(t)\right]\rho(t) = 0, \tag{3.17}$$

$$\frac{1}{2}\rho^{(7)}(t) + c\rho^{(5)}(t) + \left[3e(t) + \frac{c^2}{2}\right]\dddot{\rho} + \frac{9}{2}\dot{e}(t)\ddot{\rho}(t) + \left[2ce(t) + \frac{7}{2}\ddot{e}(t)\right]\dot{\rho}(t) + \left[\dddot{e}(t) + c\dot{e}(t)\right]\rho(t) = 0, \tag{3.18}$$

and

$$\frac{3}{2}\rho^{(5)}(t) + \frac{3c}{2}\dddot{\rho}(t) + 4e(t)\dot{\rho}(t) + 3\dot{e}(t)\rho(t) = 0. \tag{3.19}$$

It is easy to check that the function $\phi(t)$ must be identically equal to zero in order to fulfill the whole set of equations (3.14)–(3.16). Similarly, the function $\rho(t)$ is also zero if it satisfies the set of equations (3.17)–(3.19). Therefore, using the solution (3.12), the integral of motion takes the form

$$I = \int f(t)\psi(t)dt + \left[\dddot{\psi}(t) + c\dot{\psi}(t)\right]x + \left[-\ddot{\psi}(t) - c\psi(t)\right]\dot{x} + \dot{\psi}(t)\ddot{x} - \psi(t)\dddot{x}\ , \tag{3.20}$$

where x and ψ are solutions of (3.1) and (3.13), respectively. One can check easily by direct derivation that (3.20) is an integral of motion.

To investigate the noise effects on the integral of motion (3.20), we will use the same results of Sect. 2 together with the solutions of the homogeneous PUO (3.13). Note that those solutions can be written as

$$d\Psi_t = a(t, \Psi_t)dt + M(t, \Psi_t)dM_t, \tag{3.21}$$

where M_t is the normalized stochastic variable, and the vectors $d\Psi_t, a(t, \Psi_t), M(t, \Psi_t)$ are

$$\begin{cases} d\Psi_t = (d\psi, d\dot{\psi}, d\ddot{\psi}, d\dddot{\psi})^\top, \\ a(t, \Psi_t) = (\dot{\psi}, \ddot{\psi}, \dddot{\psi}, -c\ddot{\psi} - e\psi)^\top, \\ M(t, \Psi_t) = (0, 0, 0, \alpha_M\psi)^\top. \end{cases} \tag{3.22}$$

It is important to note that the multiplicative noise used in (2.2) must be the same in (3.22), in such a way that the time-dependent coefficient $e(t)$ of (3.1) and (3.13) is equal to the sum of the constant e and the corresponding multiplicative noise in each case.

We obtain the noise effects on the integral of motion (3.20) using the same results of Sect. 2 for the equation (3.1). More concretely, we solve (3.22) with the same parameters, initial conditions and multiplicative noise. The results for the homogeneous case ($f(t) = 0$) are shown in Fig. 2, and for the forced case ($f(t) = \sin^2(3t)$) in Fig. 3. We calculate the corresponding average deviation S of the noisy integral of motion respect to the non-noisy one, for all the cases used in Sect. 2. The results are shown in Table 2. One can readily see that the integral of motion is practically immune to the multiplicative noise, though it is not the case for the additive noise where the integral of motion is visibly deformed. This deformation is greater for the

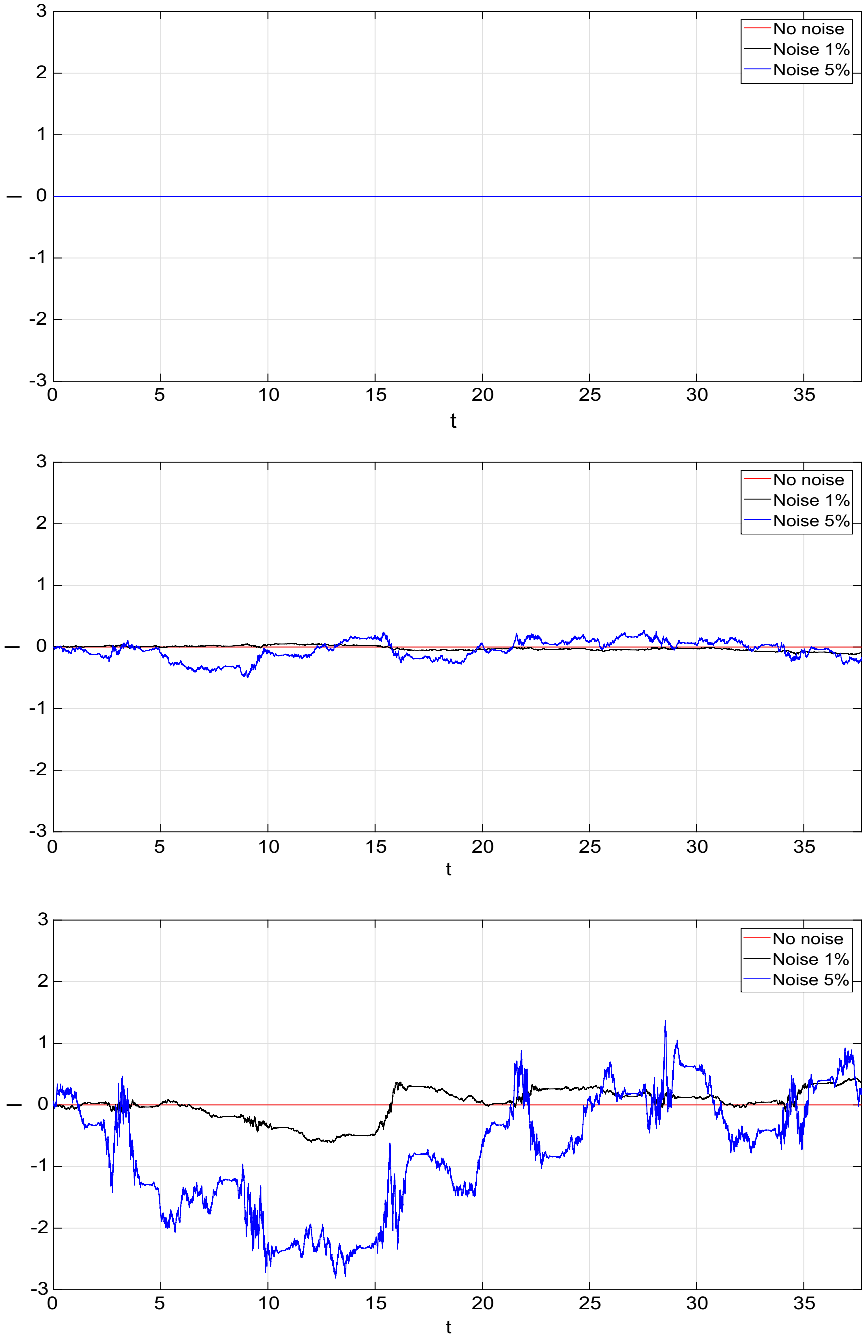

Fig. 2 Integral of motion of the homogeneous PUO without noise (red), 1% (black) and 5% (blue) of noise intensity for only multiplicative noise (up), only additive noise (middle) and simultaneous additive–multiplicative noise (down) (Color figure online)

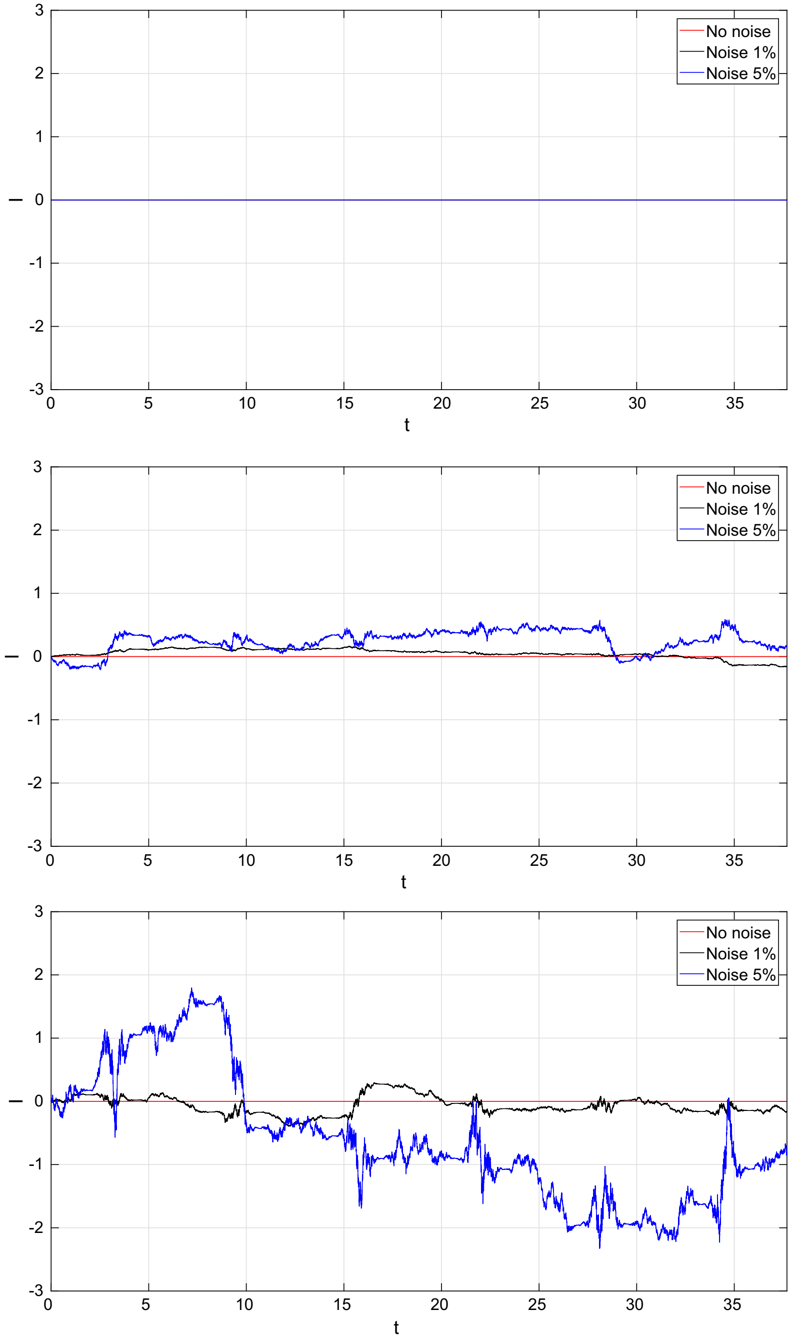

Fig. 3 Integral of motion of the forced PUO without noise (red), 1% (black) and 5% (blue) of noise intensity for only multiplicative noise (up), only additive noise (middle) and simultaneous additive–multiplicative noise (down) (Color figure online)

Table 2 Average deviations of the integral of motion for the homogeneous PUO ($f(t) = 0$) and the forced PUO ($f(t) = \sin^2(3t)$) with all possible combinations of 1% and 5% simultaneous additive–multiplicative noise

Additive noise percent (%)	Multiplicative noise percent (%)	Average deviation S
Homogeneous PUO ($f(t) = 0$)		
1	0	0.04444
5	0	0.16240
0	1	0.00000
1	1	0.25246
5	1	0.23500
0	5	0.00000
1	5	1.04940
5	5	1.22141
Forced PUO ($f(t) = \sin^2(3t)$)		
1	0	0.09075
5	0	0.30074
0	1	0.00007
1	1	0.15715
5	1	0.18914
0	5	0.00010
1	5	1.50531
5	5	1.18164

simultaneous additive–multiplicative noise. Note that similar results are obtained for the Ermakov–Lewis invariant in [16–18].

4 Noise in the PUO as a perturbation of its primary frequencies

Now, we want to include simultaneously both additive and multiplicative noises in the equation (1.1). In this case, the multiplicative case is viewed as a perturbation added to the primary frequencies ω_1^2 and ω_2^2, where $c = \omega_1^2 + \omega_2^2$ and $e = \omega_1^2\omega_2^2$. We will consider three different and independent noises, so it is convenient to write (1.1) in the form

$$dX_t = a(t, X_t)dt + A(t, X_t)dA_t + M1(t, X_t)dM1_t + M2(t, X_t)dM2_t, \quad (4.1)$$

where A_t, $M1_t$ and $M2_t$ are the normalized stochastic variables, and the vectors dX_t, $a(t, X_t)$, $A(t, X_t)$, $M1(t, X_t)$ and $M2(t, X_t)$ are:

$$\begin{cases} dX_t = (dx, d\dot{x}, d\ddot{x}, d\dddot{x})^\top, \\ a(t, X_t) = (\dot{x}, \ddot{x}, \dddot{x}, -(\omega_1^2 + \omega_2^2)\ddot{x} - (\omega_1^2\omega_2^2)x + f(t))^\top, \\ A(t, X_t) = (0, 0, 0, \alpha_A)^\top, \\ M1(t, X_t) = (0, 0, 0, \alpha_{M1}(\ddot{x} + \omega_2^2 x))^\top, \\ M2(t, X_t) = (0, 0, 0, \alpha_{M2}(\ddot{x} + \omega_1^2 x))^\top, \end{cases} \quad (4.2)$$

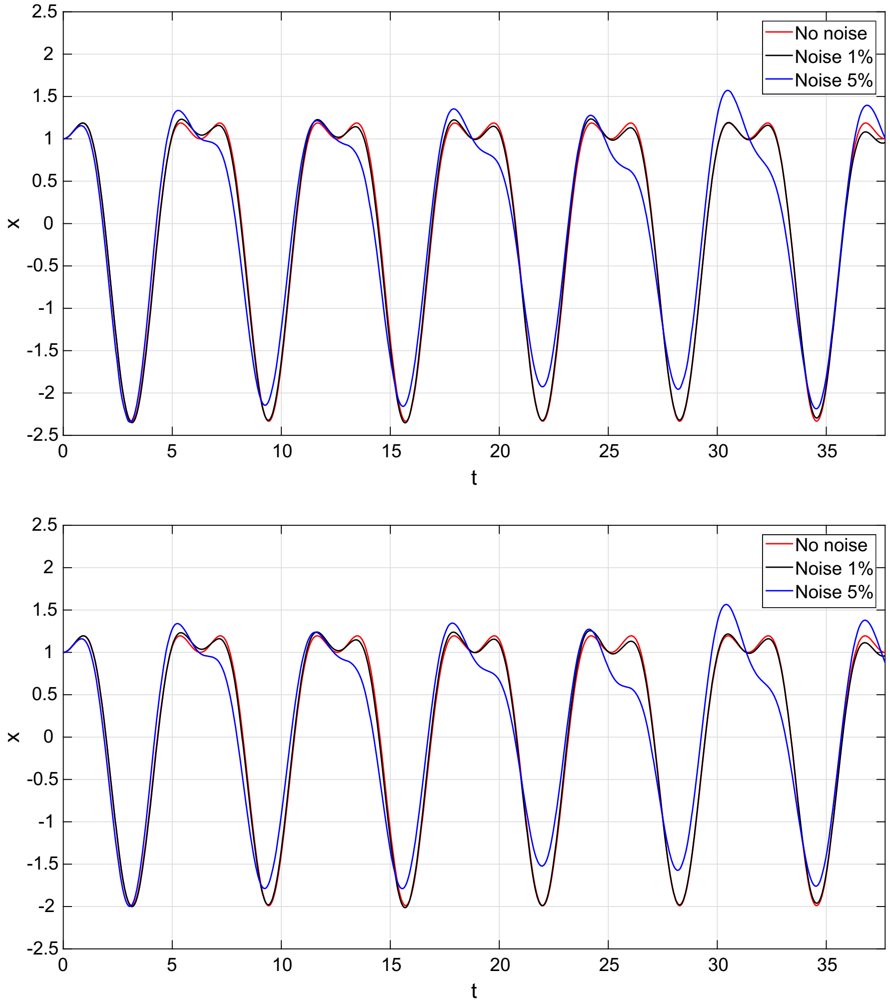

Fig. 4 Solution of the homogeneous (up) and forced (down) PUO with simultaneous additive–multiplicative noise viewed as a perturbation on primary frequencies. Red color corresponds to the solution without noise, black color corresponds to the solution with noise intensity $\alpha_A = 0.02$, $\alpha_{M1} = 0.01$, $\alpha_{M2} = 0.04$ (1%) and blue color corresponds to the solution with noise intensity $\alpha_A = 0.1$, $\alpha_{M1} = 0.05$, $\alpha_{M2} = 0.2$ (5%) (Color figure online)

where α_A is the additive noise intensity, α_{M1} (respectively, α_{M2}) is the noise intensity added to ω_1^2 (respectively, ω_2^2), and we have dropped terms of order α^2.

To obtain numerical results, we set $\omega_1^2 = 1$ and $\omega_2^2 = 4$, with initial conditions $x(0) = 1, \dot{x}(0) = 0, \ddot{x}(0) = 1$ and $\dddot{x}(0) = 0$. Similarly to previous sections, we want to study the homogeneous PUO with $f(t) = 0$ and the forced PUO with $f(t) = \sin^2(3t)$. For the intensity of additive noise we use the same $\alpha_A = 0.02, 0.1$, that correspond respectively to 1% and 5% of the amplitude of the solution (approximately). For multiplicative noise we use $\alpha_{M1} = 0.01, 0.05$ that correspond respectively to 1%

Table 3 Average deviations of the homogeneous PUO with all possible combinations of 1% and 5% simultaneous additive–multiplicative noise viewed as a perturbation on primary frequencies

Additive noise percent (%)	ω_1^2 noise percent (%)	ω_2^2 noise percent (%)	Average deviation S
1	0	0	0.01195
5	0	0	0.04536
0	0	1	0.02913
1	0	1	0.03223
5	0	1	0.11696
0	0	5	0.27124
1	0	5	0.18168
5	0	5	0.19914
0	1	0	0.02868
1	1	0	0.02305
5	1	0	0.07743
0	1	1	0.03844
1	1	1	0.04370
5	1	1	0.10593
0	1	5	0.19783
1	1	5	0.25317
5	1	5	0.23996
0	5	0	0.11812
1	5	0	0.11805
5	5	0	0.12044
0	5	1	0.15513
1	5	1	0.13622
5	5	1	0.12246
0	5	5	0.21587
1	5	5	0.26883
5	5	5	0.28908

and 5% of the parameter ω_1^2, and $\alpha_{M2} = 0.04, 0.2$ that corresponds to 1% and 5%, respectively, of the parameter ω_2^2.

Again, we solve the equation (4.1) using the fourth order Runge–Kutta method [20], and we generate the normalized stochastic variable using the Box-Müller algorithm [21]. The results for simultaneous additive–multiplicative 1% and 5% noise for homogeneous PUO and forced PUO are shown in Fig. 4. As in Sect. 2, we obtain the corresponding average deviation S of the noisy solutions respect the solution without noise. The results are shown in Tables 3 and 4.

We can see more noise influence as the percentage increases. Similarly to Sect. 2, we note that the additive noise affects similarly to the homogeneous and forced PUO, but the multiplicative noise perturbs the solutions more in both cases (see for example rows 2, 6, 18 and 24 in Tables 3 and 4). In addition, note that a perturbation on the larger frequency ω_2^2 is more significant than a perturbation on the smaller frequency ω_1^2.

Table 4 Average deviations of the forced PUO with all possible combinations of 1% and 5% simultaneous additive–multiplicative noise viewed as a perturbation on primary frequencies

Additive noise percent (%)	ω_1^2 noise percent (%)	ω_2^2 noise percent (%)	Average deviation S
1	0	0	0.01124
5	0	0	0.04536
0	0	1	0.05938
1	0	1	0.02876
5	0	1	0.05854
0	0	5	0.16342
1	0	5	0.16354
5	0	5	0.23770
0	1	0	0.02888
1	1	0	0.02239
5	1	0	0.08771
0	1	1	0.04789
1	1	1	0.04400
5	1	1	0.10334
0	1	5	0.18205
1	1	5	0.11520
5	1	5	0.35892
0	5	0	0.09631
1	5	0	0.11410
5	5	0	0.15600
0	5	1	0.07883
1	5	1	0.16571
5	5	1	0.12452
0	5	5	0.22863
1	5	5	0.24910
5	5	5	0.31332

5 Discussions and conclusion

We have included simultaneously additive and multiplicative noise to the PUO, homogeneous and forced, with different intensities. Also, we investigate the noise effects on an integral of motion of the PUO. After that, we have included noise as a perturbation on the primary frequencies of the oscillator because the PUO can be viewed as two coupled harmonic oscillators. After the numerical calculation we found that:

- The solution is more deformed with more noise intensity.
- The multiplicative noise affects more the solutions than the additive one.
- The integral of motion is practically immune to the multiplicative noise.
- The simultaneous noise increases more the deformation of the integral of motion than individual noises.

– Respect to the solutions, there is no significant difference between the average deviation of the homogeneous and forced PUO.
– A perturbation on the larger frequency produces a larger effect in the solution.

The integral of motion is no affected for the multiplicative noise because by construction, this integral considers the multiplicative noise as part of the internal system. The last sentence can be interpreted as if we have two coupled harmonic oscillators, any perturbation on the oscillator with larger frequency will affect to a greater extent to the whole system.

As conclusion of this work, we have implemented simultaneously additive and multiplicative noise to the PUO, homogeneous and forced. We found that the solution is more sensitive to the multiplicative noise. We construct an integral of motion for the PUO when one coefficient is time dependent and we found this integral of motion is highly immune to the multiplicative noise. Also, we investigate what happens if we add noise to the primary frequencies of the oscillator if it is viewed as two coupled harmonic oscillators. Considering that the PUO can be viewed as a two coupled harmonic oscillators system, we think the results presented in this work can be useful to investigate the corresponding noisy systems.

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