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Driven damped *n*th-power anharmonic oscillators with time-dependent coefficients and their integrals of motion

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ABSTRACT

In this manuscript, we derive integrals of motion for general anharmonic oscillators with damping and power-law forcing. The model under investigation has time-dependent coefficients, and the determination of these physical quantities is carried out using Noether's theorem. The solutions must satisfy appropriate analytical conditions for the proposed quantities to be true integrals of motion. In turn, these analytical conditions are associated to well known physical systems, including the Milne-Pinney and Ermakov-Lewis models. We provide various numerical solutions of our equations of motion and the associated integrals to verify the theoretical results.

1. Introduction

Systems of nonlinear oscillators constitute some of the most important problems in mathematical physics [1] in view of their many potential applications to nonlinear physics [2–4]. One of the most notable nonlinear oscillators was investigated by G. Duffing (1918) to study the dynamics of oscillatory phenomena in physics [5]. Moreover, a generalization of this nonlinear oscillator is found in the form of the Emden-Fowler equation (see [6] and references therein). As expected, the determination of exact analytical solutions for these and other nonlinear oscillators is a very difficult task in general, especially for non-autonomous systems (that is, systems in which the coefficients are dependent on time). It is worth mentioning that some of the most important time-dependent systems with applications in vibration mechanics, electromagnetism and particle physics are the so-called Mathieu-type equations [7].

From the analytical point of view, the dynamics of a nonlinear system can be better understood using integrals of motion. Indeed, integrals of motion are usually employed to gather information about some important properties of physical systems [8]. Moreover, the resolution of some equations of motion may be simplified substantially if an integral of motion is determined. Indeed, these quantities are used to investigate analytical features of the solutions without solving the equations of motion. Alternatively, these quantities are employed to build Lyapunov functionals applying the Chetayev approach [9]. In fact, it is well known that Lyapunov functions provide information on the stability properties of the solution using the method of Lyapunov [10]. Alternatively, some analytical techniques have been employed to solve nonlinear systems, including the homotopy perturbation method for systems with variable coefficients [11] and the exp-function method [12].

The energy associated with a Hamiltonian in models with constant coefficients is an integral of motion. However, if the coefficients are time-dependent, then the determination of the expression of the integrals of motion may be an extremely difficult task. To alleviate this problem, the Ermakov-Lewis invariants were derived for temporally varying harmonic oscillators [13,14]. In [15], several generalizations of the invariants for these systems were introduced. Moreover diverse applications of these integrals in quantum mechanics and cosmology were disused in [16,17]. For more complex models, these physical quantities may be calculated by conditioning the solutions [18,19]. In particular, an integral of motion for a Duffing model with cubic-quintic

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forcing and non-autonomous regime was obtained in [20] for equations with nth power nonlinearities [21]. It is worthwhile to recall that some analytical techniques to obtain these physical quantities are based on transformation groups [22], dynamic and algebraic arguments [23,24] or Noether's theorem [25,26].

The main goal of the present manuscript is to derive the integrals of motion for a forced and damped nth anharmonic oscillator with variable coefficients. To that end, our study will hinge on the application of Noether's theorem. The remainder of this paper is arranged as follows. In Section 2, we obtain integrals of motion for nth anharmonic oscillators with time-dependent coefficients and the associated constraints that guarantee the existence of the solutions. Section 3 includes numerical examples to verify the theoretical findings. The computational tests prove that the physical quantities derived in this work are constant with respect to time, in agreement with the theoretical results.

2. Integrals of motion

The present manuscript employs the analytic formalism introduced by Lutzky in [25]. It is worth pointing out that this formalism is based on Noether's theorem [26], and it is described in various papers available in the literature (see [20]). Concretely, we will employ the fact that the integral of motion corresponding to Lagrange's functional $L(x,\dot{x},t)$ is readily derived if the transformation defined by means of the symmetry group operator is action-invariant. In fact, that physical quantity can be expressed by means of the formula

$$I = F - \xi L - (\eta - \xi\dot{x})\frac{\partial L}{\partial \dot{x}}. \tag{1}$$

In this identity, $F(x,t)$ is obtained by adding the derivative to L before our transformation, and the following analytical constraint must be satisfied:

$$\dot{F}(x,t) = \dot{\xi}L + \eta\frac{\partial L}{\partial x} + \xi\frac{\partial L}{\partial t} + (\dot{\eta} - \dot{x}\dot{\xi})\frac{\partial L}{\partial \dot{x}}. \tag{2}$$

Here, $\dot{\eta}, \dot{\xi}$ and $\dot{F}$ are defined by

$$\begin{cases} \dot{\eta} = \dot{x}\dfrac{\partial \eta}{\partial x} + \dfrac{\partial \eta}{\partial t}, \\ \dot{\xi} = \dot{x}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial \xi}{\partial t}, \\ \dot{F} = \dot{x}\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial t}. \end{cases} \tag{3}$$

2.1. Forced nth anharmonic oscillator

We will consider firstly a forced anharmonic oscillator without damping governed by an nth power-law, and we will suppose that the coefficients are time-dependent. More concretely, let us consider the model

$$\frac{d^2x}{dt^2} + a_1(t)x + a_n(t)x^n = f(t). \tag{4}$$

Notice that the Lagrangian L associated with the Eq. (4) is given by

$$L = \begin{cases} \dfrac{1}{2}\left[\dot{x}^2 - a_1(t)x^2 - \dfrac{2}{n+1}a_n(t)x^{n+1} + 2f(t)x\right], & \text{for } n \neq -1, \\ \dfrac{1}{2}\left[\dot{x}^2 - a_1(t)x^2 + 2f(t)x - 2a_n(t)\ln|x|\right], & \text{for } n = -1. \end{cases} \tag{5}$$

Applying the Euler-Lagrange equation to the Lagrangian (5), the equation of motion (4) is readily obtained. Substituting this Lagrangian into condition (2) together with the expressions for $\dot{F}$ and $\dot{\eta}$ defined in 3, we obtain the following polynomial on x and $\dot{x}$, for $n \neq -1$:

$$\begin{aligned} &\left[\frac{\partial \eta}{\partial x} - \frac{1}{2}\dot{\xi}\right]\dot{x}^2 + \left[\frac{\partial \eta}{\partial t} - \frac{\partial F}{\partial x}\right]\dot{x} + \left[-\frac{1}{n+1}\dot{a}_n(t)\xi \right. \\ &\left. - \frac{1}{n+1}a_n(t)\dot{\xi}\right]x^{n+1} - a_n(t)\eta x^n + \left[-\frac{1}{2}\dot{a}_1(t)\xi - \frac{1}{2}a_1(t)\dot{\xi}\right]x^2 + x\left[\xi\dot{f}(t) \right. \\ &\left. - \eta a_1(t) + f(t)\dot{\xi}\right] + \left[\eta f(t) - \frac{\partial F}{\partial t}\right] \\ &= 0. \end{aligned} \tag{6}$$

Using linear independence, each of the expressions of (6) in brackets must be equal to zero, and so must be the factors of $\dot{x}^2$ and $\dot{x}$. Thus, the following parametric functions are obtained:

$$\eta = \psi + \frac{1}{2}\dot{\xi}x, \tag{7}$$

$$F = \dot{\psi}x + \frac{1}{4}\ddot{\xi}x^2 + \chi. \tag{8}$$

In these expressions, $\chi(t)$ and $\psi(t)$ are unknown functions. Substituting these new functions into (6), the following equation is readily reached:

$$\begin{aligned} &\left[-\frac{1}{n+1}\dot{a}_n(t)\xi - \frac{n+3}{2(n+1)}a_n(t)\dot{\xi}\right]x^{n+1} - a_n(t)\psi(t)x^n + \left[-\frac{1}{2}\dot{a}_1(t)\xi - a_1(t)\dot{\xi} \right. \\ &\left. - \frac{1}{4}\dddot{\xi}\right]x^2 + x\left[\frac{3}{2}\dot{\xi}f(t) + \dot{f}(t)\xi - \ddot{\psi}(t) - a_1(t)\psi(t)\right] - [\dot{\chi}(t) - f(t)\psi(t)] \\ &= 0. \end{aligned} \tag{9}$$

The coefficient of x^n and the last bracket of 9 indicate that $\psi = 0$, and that there exists a constant $C \in \mathbb{R}$ with the property that $\chi = C$. In this work, we assume that $C = 0$ and $\xi(t) = \rho^2$. The other coefficients of 9 satisfy the conditions

$$a_n(t) = \frac{C_n}{\rho^{n+3}}, \qquad f(t) = \frac{C_f}{\rho^3}, \qquad \ddot{\rho} + a_1(t)\rho = \frac{K}{\rho^3}, \tag{10}$$

where C_n, C_f and K are real numbers. Notice that the Milne-Pinney equation [27,28] is obtained from the last condition of 10. Substituting all these expressions into (1) shows that

$$I = \frac{1}{2}(\dot{\rho}(t)x - \rho(t)\dot{x})^2 + \frac{1}{1+n}a_n(t)\rho^2x^{n+1} - f(t)\rho^2(t)x + \frac{K}{2}\left[\frac{x}{\rho}\right]^2, \quad \text{for} n \neq -1. \tag{11}$$

Finally, differentiating directly confirms that (11) is indeed an integral of motion.

If we perform the same procedure for the Case $n = -1$, then we reach the integral of motion

$$I = \frac{1}{2}(\dot{\rho}(t)x - \dot{x}\rho(t))^2 + C_{-1}\ln\left|\frac{x}{\rho}\right| - f(t)x\rho^2(t) + \frac{K}{2}\left[\frac{x}{\rho}\right]^2, \tag{12}$$

where $C_{-1} \in \mathbb{R}$ is an arbitrary constant, whereas $a_{-1}(t), \rho(t)$ and $f(t)$ satisfy the same conditions (10). However, notice that if we consider an oscillator of the form

$$\frac{d^2x}{dt^2} + a_1(t)x + \sum_{i=-mi\neq 1}^{n} a_i(t)x^i = f(t), \tag{13}$$

and if we apply the previous results, then we reach the integral of motion

$$I = \frac{1}{2}(\rho(t)\dot{x} - \dot{\rho}(t)x)^2 + \sum_{i=-mi\neq -1,1}^{n} \frac{1}{i+1}a_i(t)\rho^2x^{i+1} + C_{-1}\ln\left|\frac{x}{\rho}\right| - f(t)\rho^2(t)x + \frac{K}{2}\left[\frac{x}{\rho}\right]^2, \tag{14}$$

where the functions $a_i(t)$ satisfy the conditions (10).

2.2. Damped nth anharmonic oscillator

Let us consider the damped nth anharmonic oscillator governed by

$$\frac{d^2x}{dt^2}+b(t)\frac{dx}{dt}+a_1(t)x+\sum_{i=-mi\neq 1}^{n}a_i(t)x^i=f(t). \tag{15}$$

To derive the integral of motion associated with the damped nth anharmonic oscillator (15), the following transformation is used (see [18]):

$$y(t)=\exp\left(\int^t\frac{b(t)}{2}dt\right)x(t). \tag{16}$$

As a consequence, the resulting equation of motion is given by

$$\ddot{y}+\left(-\frac{1}{4}b^2(t)-\frac{1}{2}\dot{b}(t)+a_1(t)\right)y+\sum_{i=-mi\neq 1}^{n}\left(a_i(t)e^{\frac{1-i}{2}\int^t b(t)dt}\right)y^i$$
$$=f(t)e^{\frac{1}{2}\int^t b(t)dt}. \tag{17}$$

Using the transformation 16, the Eq. (17) assumes the form of an undamped equation of motion like (13). Thus, the associated integral takes the form

$$I_y=\frac{1}{2}(\rho_y\dot{y}-\dot{\rho}_y y)^2+\sum_{i=-mi\neq -1,1}^{n}\frac{1}{i+1}a_{iy}\rho_y^2y^{i+1}+C_{-1y}\ln\left|\frac{y}{\rho_y}\right|-f_y\rho_y^2y+\frac{K_y}{2}\left[\frac{y}{\rho_y}\right]^2, \tag{18}$$

where

$$a_{iy}(t)=a_i(t)e^{\frac{1-i}{2}\int^t b(t)dt}, \tag{19}$$

$$f_y(t)=f(t)e^{\frac{1}{2}\int^t b(t)dt}. \tag{20}$$

In this case, the restrictions for the real numbers C_{iy}, C_{fy} and K_y are

$$a_{iy}(t)=\frac{C_{iy}}{\rho_y^{i+3}},\qquad f_y(t)=\frac{C_{fy}}{\rho_y^3},\qquad \dot{\rho}_y+\left(a_1(t)-\frac{1}{4}b^2(t)-\frac{1}{2}\dot{b}(t)\right)\rho_y=\frac{K_y}{\rho_y^3}. \tag{21}$$

Using the solutions and parameters of the model (15), we can rewrite (18) in the following way:

$$I_y=e^{\int^t b(t)dt}\left[\frac{1}{2}\left(\rho_y\dot{x}-\dot{\rho}_y x+\frac{1}{2}b(t)\rho_y x\right)^2+\sum_{i=-mi\neq -1,1}^{n}\frac{1}{i+1}a_i(t)\rho_y^2x^{i+1}\right.$$
$$\left.+a_{-1}(t)\rho_y^2\ln\left|\frac{xe^{\frac{1}{2}\int^t b(t)dt}}{\rho_y}\right|-f(t)\rho_y^2x+\frac{K_y}{2}\left[\frac{x}{\rho_y}\right]^2\right]. \tag{22}$$

Again, it can be proved that (22) is an integral of motion by directly differentiating.

3. Numerical implementation

Some computer experiments are presented now to evaluate the validity of the results in the previous section. To that end, we will require that the conditions (10) and (21) be satisfied. As mentioned above, these conditions are directly related to the Milne-Pinney equation, which is a model for which there are exact solutions available (see [29]). More precisely, in the ordinary differential equation

$$\frac{d^2z}{dt^2}+Q^2(t)z=0, \tag{23}$$

the coefficient $Q^2(t)$ corresponds to a_1 for the conditions of the Milne-Pinney model associated with our motion integral of undamped system (10). In the Case of the damped oscillator (21), that coefficient corresponds to the coefficient of ρ_y. In the following experiments, the initial conditions will be $x(0)=\rho(0)=1$ and zero initial velocities. We will consider the next cases in our examples below (similar examples were considered in [20,30]):

- **Case 1** $\begin{cases} Q(t)=\frac{1}{4}, \\ \dot{\rho}+\frac{1}{16}\rho=\frac{1}{\rho^3}, \\ \rho(t)=\sqrt{1+15\sin^2\left(\frac{t}{4}\right)}. \end{cases}$
- **Case 2** $\begin{cases} Q(t)=\frac{1}{4}\sin t, \\ \dot{\rho}+\frac{1}{16}\sin^2(t)\rho=\frac{1}{\rho^3}, \\ \rho(t)=\sqrt{\left(\frac{C\left(\frac{1}{32},\frac{1}{64};t\right)}{C\left(\frac{1}{32},\frac{1}{64};0\right)}\right)^2+\left(\frac{S\left(\frac{1}{32},\frac{1}{64};t\right)}{\dot{S}\left(\frac{1}{32},\frac{1}{64};0\right)}\right)^2}. \end{cases}$

Here, $C\left(\frac{1}{32},\frac{1}{64};0\right)\approx 0.9919$ and $\dot{S}\left(\frac{1}{32},\frac{1}{64};0\right)\approx 0.1785$, where S and C denote, respectively, the well-known Mathieu sine and cosine functions. In Case 1, the frequency $Q(t)$ represents the natural frequency, and it represents the frequency associated to an equation of the Mathieu-type in Case 2. For the remainder of this section, we will use these solutions to approximate computationally the solutions of the equations of motion and their associated integrals of motion through a fourth-order Runge-Kutta scheme.

Example 1. (*Model with x^{-1} and without damping*) Fix the constraints (10) with $C_{-1}=\frac{1}{16}, C_f=1$ and $K=1$. The corresponding solutions for the Milne-Pinney model are the following:

- Case 1 $\begin{cases} Q(t)=\frac{1}{4}, \\ \rho^{-3}=\ddot{x}+\frac{1}{16}x+\frac{1}{16}\rho^{-2}x^{-1}. \end{cases}$
- Case 2 $\begin{cases} Q(t)=\frac{1}{4}\sin(t), \\ \rho^{-3}=\dot{x}+\frac{1}{16}\sin^2(t)x+\frac{1}{16\rho^2x}. \end{cases}$

The graphs on the left of Fig. 1 provide approximations to the solutions of the anharmonic oscillator 4 with $n=-1$ and without damping. Meanwhile, the graphs on the right depict the associated integrals. These graphs confirm that the corresponding integrals of motion remains constant over time.

Example 2. (*Model with x^{-1} and damping*) In this case, we will employ the set of constraints (21) with parameters $C_{-1y}=\frac{1}{16}, C_{fy}=1$, $K_y=1$ and $b(t)=\frac{1}{16}$.

- Case 1 $\begin{cases} Q(t)=\frac{1}{4}, \\ e^{-\frac{t}{32}}\rho_y^{-3}=\ddot{x}+\frac{1}{16}\dot{x}+\frac{65}{1024}x+\frac{1}{16}e^{-\frac{t}{16}}\rho_y^{-2}x^{-1}. \end{cases}$

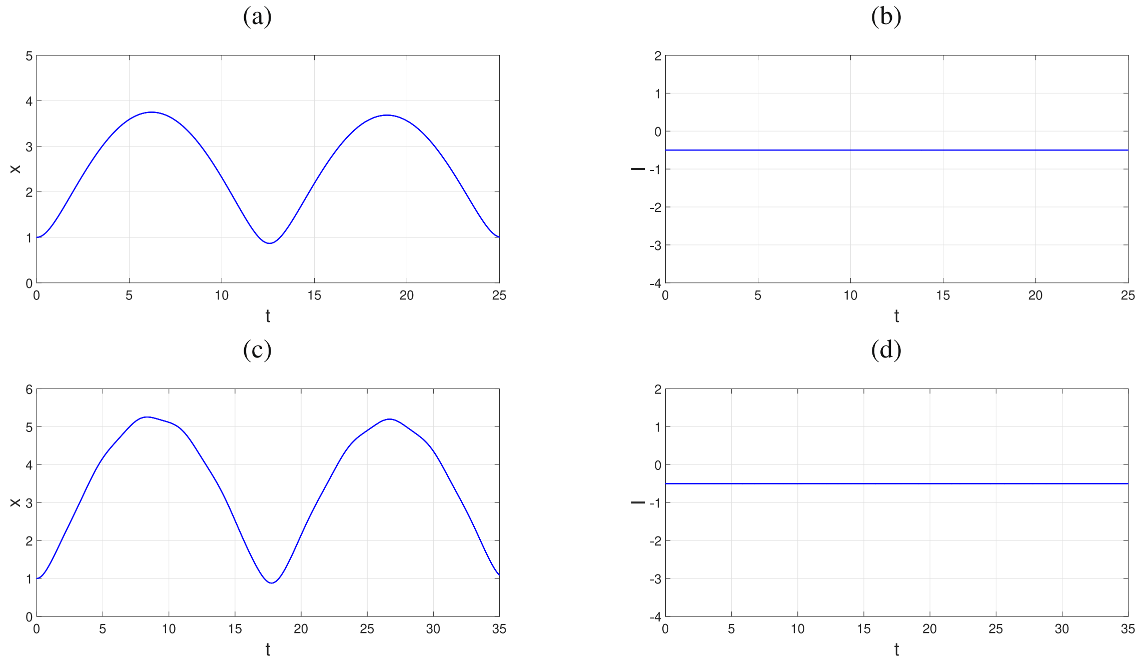

Fig. 1. Undamped x^{-1} anharmonic oscillator solution (left) and its correspondence integral of motion (right). In these simulations, we have employed the functions $Q(t) = \frac{1}{4}$ (top), $Q(t) = \frac{1}{4}\sin t$ (bottom). We considered two cases in these simulations, namely, **Case 1** (top row) and **Case 2** (bottom row), using the constraints (10) with $C_{-1} = \frac{1}{16}, C_f = 1$ and $K = 1$. The initial data are $x(0) = \rho(0) = 1$ and zero initial velocities..

- Case 2 $$\begin{cases} Q(t) = \dfrac{1}{4}\sin(t), \\ e^{-\frac{t}{32}}\rho_y^{-3} = \ddot{x} + \dfrac{1}{16}\dot{x} + \dfrac{1}{16}\left[\dfrac{1}{64} + \sin^2(t)\right]x + \dfrac{1}{16}e^{-\frac{t}{16}}\rho_y^{-2}x^{-1}. \end{cases}$$

Fig. 2 illustrates the results associated with the damped x^{-1} model and the respect$C_{-2} = C_{-1} = C_3 = C_5 = \frac{1}{16}, C_f = 1, K = 1$ive integral. The results confirm that these quantities are temporally invariant.

Example 3. (*Model with x^{-2} and without damping*) We employ the constraints (10), letting $C_{-2} = \frac{1}{16}, C_f = 1$ and $K = 1$. Under those circumstances, we obtain:

- Case 1 $$\begin{cases} Q(t) = \dfrac{1}{4}, \\ \rho^{-3} = \ddot{x} + \dfrac{1}{16}x + \dfrac{1}{16}\rho^{-1}x^{-2}. \end{cases}$$

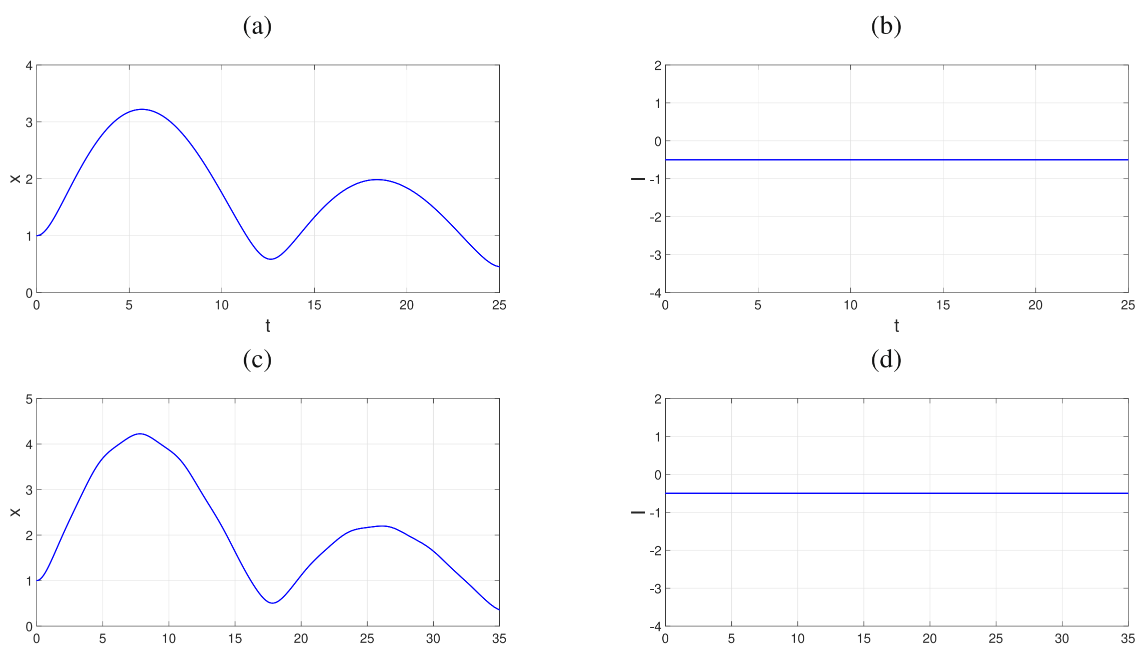

Fig. 2. Damped x^{-1} anharmonic oscillator solution (left) and the associated integral of motion (right). In these computer experiments, we defined $b(t) = \frac{1}{16}$, and let $Q(t) = \frac{1}{4}$ (top), $Q(t) = \frac{1}{4}\sin t$ (bottom). We considered two cases in these simulations, namely, **Case 1** (top row) and **Case 2** (bottom row), using the constraints (21) with $C_{-1y} = \frac{1}{16}, C_{fy} = 1, K_y = 1$ and $b(t) = \frac{1}{16}$. The initial data are $x(0) = \rho(0) = 1$ and zero initial velocities..

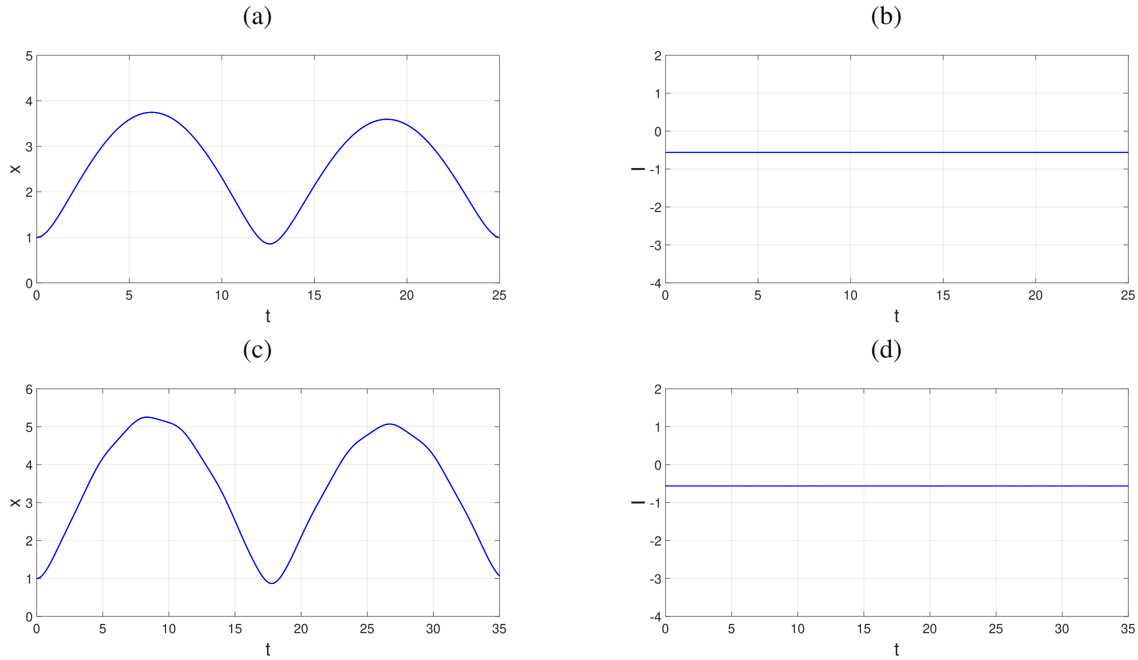

Fig. 3. Undamped x^{-2} anharmonic oscillator solution (left) and the associated integral of motion (right). In these experiments, we employed $Q(t) = \frac{1}{4}$ (top), $Q(t) = \frac{1}{4}\sin t$ (bottom). We considered two cases in these simulations, namely, **Case 1** (top row) and **Case 2** (bottom row), using the constraints (10) with $C_{-2} = \frac{1}{16}, C_f = 1$ and $K = 1$. The initial data are $x(0) = \rho(0) = 1$ and zero initial velocities..

- Case 2 $\begin{cases} Q(t) = \dfrac{1}{4}\sin(t), \\ \rho^{-3} = \ddot{x} + \dfrac{1}{16}\sin^2(t)x + \dfrac{1}{16}\rho^{-1}x^{-2}. \end{cases}$

The left column of Fig. 3 provides the results corresponding to the model 4 and the cases considered in this work. Meanwhile, the associated integrals of motion appear on the right column. Our findings confirm that these physical quantities remain constant over time.

Example 4. (*Damped x^{-2} anharmonic oscillator*) Let the constraints (21) be given by $C_{-2y} = \frac{1}{16}, C_{fy} = 1, K_y = 1$ and $b(t) = \frac{1}{16}$. We have the following results under these circumstances:

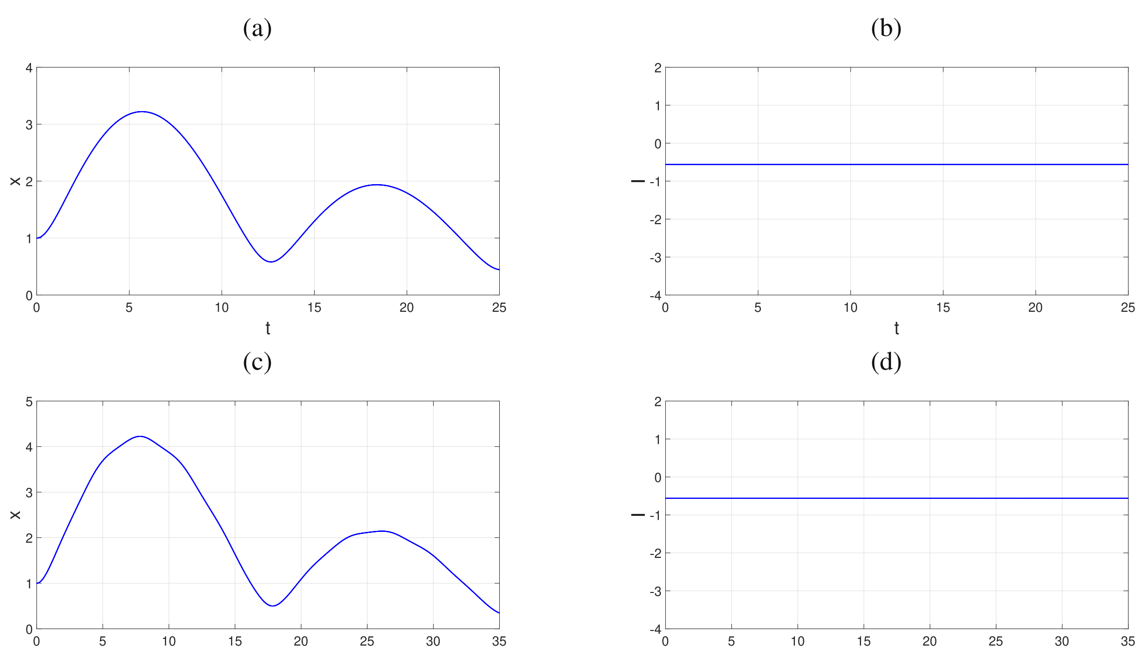

Fig. 4. Damped x^{-2} anharmonic oscillator solution (left) and the associated integral of motion (right). For these computer simulations, we employed $b(t) = \frac{1}{16}$ along with $Q(t) = \frac{1}{4}$ (top), $Q(t) = \frac{1}{4}\sin t$ (bottom). We considered two cases in these simulations, namely, **Case 1** (top row) and **Case 2** (bottom row), using the constraints (21) with $C_{-2y} = \frac{1}{16}, C_{fy} = 1, K_y = 1$ and $b(t) = \frac{1}{16}$. The initial data are $x(0) = \rho(0) = 1$ and zero initial velocities..

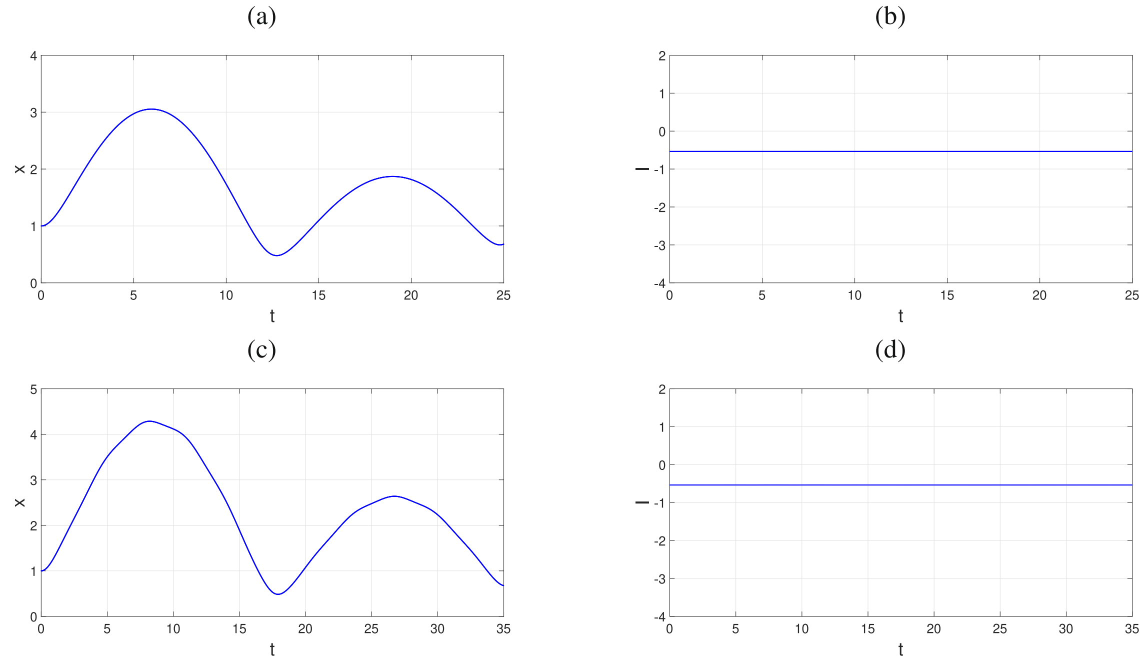

Fig. 5. Undamped multi-anharmonic oscillator solution (left) and the associated integral of motion (right). In these computer experiments, we set $Q(t)=\frac{1}{4}$ (top) and $Q(t)=\frac{1}{4}\sin t$ (bottom). We considered two cases in these simulations, namely, **Case 1** (top row) and **Case 2** (bottom row), using the constraints (10) with $C_{-2}=C_{-1}=C_3=C_5=\frac{1}{16}, C_f=1, K=1$. The initial data are $x(0)=\rho(0)=1$ and zero initial velocities..

- Case 1
$$\begin{cases} Q(t)=\dfrac{1}{4}, \\ e^{-\frac{t}{32}}\rho_y^{-3}=\ddot{x}+\dfrac{1}{16}\dot{x}+\dfrac{65}{1024}x+\dfrac{1}{16}e^{-\frac{3t}{32}}\rho_y^{-1}x^{-2}. \end{cases}$$
- Case 2
$$\begin{cases} Q(t)=\dfrac{1}{4}\sin(t), \\ e^{-\frac{t}{32}}\rho_y^{-3}=\ddot{x}+\dfrac{1}{16}\dot{x}+\dfrac{1}{16}\left[\dfrac{1}{64}+\sin^2(t)\right]x+\dfrac{1}{16}e^{-\frac{3t}{32}}\rho_y^{-1}x^{-2}. \end{cases}$$

Fig. 4 illustrates the results associated with the damped x^{-2} anharmonic model and the respective integrals of motion. We confirm again that these quantities are temporal invariants.

Example 5. (*Multi-anharmonic oscillator without damping*) Finally, we present a multi-anharmonic oscillator which includes powers $-2,-1,3$ and 5 of x. Assume that the conditions (10) are satisfied with $C_{-2}=C_{-1}=C_3=C_5=\frac{1}{16}, C_f=1, K=1$. The two cases on the Milne-Pinney equation are described by

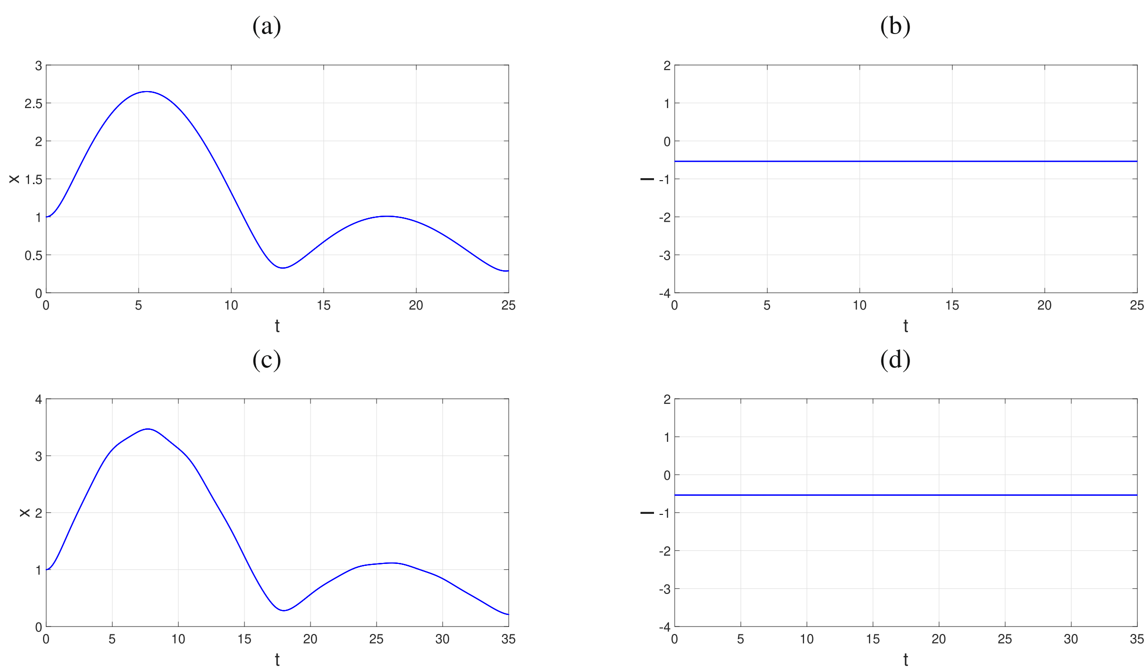

Fig. 6. Solution of the damped multi-anharmonic oscillator (left) and its associated integral of motion (right). For these simulations, we let $b(t)=\frac{1}{16}$, and we used $Q(t)=\frac{1}{4}$ (top), $Q(t)=\frac{1}{4}\sin t$ (bottom). We considered two cases in these simulations, namely, **Case 1** (top row) and **Case 2** (bottom row), using the constraints (21) with $C_{-2y}=C_{-1y}=C_{3y}=C_{5y}=\frac{1}{16}, C_{fy}=1, K_y=1$ and $b(t)=\frac{1}{16}$. The initial data are $x(0)=\rho(0)=1$ and zero initial velocities..

- Case 1 $\begin{cases} Q(t)=\frac{1}{4}, \\ \rho^{-3}=\ddot{x}+\frac{1}{16}x+\frac{1}{16}\rho^{-1}x^{-2}+\frac{1}{16}\rho^{-2}x^{-1}+\frac{1}{16}\rho^{-6}x^{3}+\frac{1}{16}\rho^{-8}x^{5}. \end{cases}$
- Case 2 $\begin{cases} Q(t)=\frac{1}{4}\sin(t), \\ \rho^{-3}=\ddot{x}+\frac{1}{16}\sin^2(t)x+\frac{1}{16}\rho^{-1}x^{-2}+\frac{1}{16}\rho^{-2}x^{-1}+ \end{cases}$

$\frac{1}{16}\rho^{-6}x^{3}+\frac{1}{16}\rho^{-8}x^{5}.$

The left column of Fig. 5 provides the approximations to the solutions of 13 without damping. The associated integrals of motion are presented on the right column. We see once more that these quantities are constant with respect to time, as expected.

Example 6. (*Multi-anharmonic model with damping*) Let us consider the constraints (21), assuming now that $C_{-2y}=C_{-1y}=C_{3y}=C_{5y}=\frac{1}{16}$, $C_{fy}=1, K_y=1$ and $b(t)=\frac{1}{16}$.

- Case 1 $\begin{cases} Q(t)=\frac{1}{4}, \\ e^{-\frac{t}{32}}\rho_y^{-3}=\ddot{x}+\frac{1}{16}\dot{x}+\frac{65}{1024}x+\frac{1}{16}e^{-\frac{3t}{32}}\rho_y^{-1}x^{-2}+\frac{1}{16}e^{-\frac{t}{16}}\rho_y^{-2}x^{-1}+ \end{cases}$

$\frac{1}{16}e^{\frac{t}{16}}\rho_y^{-6}x^{3}+\frac{1}{16}e^{\frac{t}{8}}\rho_y^{-8}x^{5}.$

- Case 2 $\begin{cases} Q(t)=\frac{1}{4}\sin(t), \\ e^{-\frac{t}{32}}\rho_y^{-3}=\ddot{x}+\frac{1}{16}\dot{x}+\frac{1}{16}\left[\frac{1}{64}+\sin^2(t)\right]x+\frac{1}{16}e^{-\frac{3t}{32}}\rho_y^{-1}x^{-2}+ \end{cases}$

$\frac{1}{16}e^{-\frac{t}{16}}\rho_y^{-2}x^{-1}+\frac{1}{16}e^{\frac{t}{16}}\rho_y^{-6}x^{3}+\frac{1}{16}e^{\frac{t}{8}}\rho_y^{-8}x^{5}.$

The results of our simulations are provided in Fig. 6, and they confirm our theoretical results again.

In the examples above, we considered various cases in order to confirm the theoretical results from Section 2. indeed, we considered examples in which damping was present and others in which it was absent. We also considered systems in which the forcing considered a single anharmonic term, and systems in which various anharmonic terms are present. Finally, various power-laws were studied, including negative and positive power-laws. In all of them, the numerical experiments showed that the integral of motion was indeed a constant quantity associated to all those systems. Obviously, these results confirmed the theoretical derivations from Section 2. On the other hand, it is important to notice that the quantitative behavior of the integrals of motion is strongly dependent on the type of non-linearity. At the same time, this qualitative behavior is independent of the expressions of the coefficients. Unfortunately, the restrictive nature of the coefficients does not allow to analyze a wider variety of cases. This is due to the fact that exact solutions are not available in closed form for arbitrary initial data. It is worth pointing out that the damping terms clearly fulfill their function in each case. All of these experiments serve to emphasize that the corresponding value of the integral of motion for each of the cases considered is determined by the initial conditions of the problem. The simulations considered in this work show that, independently of the degree of the nonlinearity and the presence of constant or non-constant coefficients, the integrals of motion derived in this work do not depend on time.

4. Conclusion

In the present work, we obtained physical quantities for general forced *n*th power anharmonic oscillators with damping term. The physical system under study considers time-dependent coefficients. These physical quantities are integrals of motion, and their calculation is carried out using Noether's theorem. It is worth pointing out that the solutions must satisfy appropriate analytical conditions for the proposed quantities to be actual integrals of motion. In turn, these analytical conditions are associated to well known physical systems, including the Milne-Pinney and Ermakov-Lewis models. We provide sufficient numerical solutions of our equations of motion and the associated integrals of motion to verify the theoretical results. As a follow-up of this work, various interesting applications are expected within the field of nonlinear analysis and its applications. As an example, the authors are interested in the investigation of nonlinear phenomena (like the nonlinear processes of supratransmission, infratransmission and bistability of energy) in systems of anharmonic oscillators with time-dependent coefficients. This phenomenon consists in the sudden increase in the amplitude of transmitted wave signals in nonlinear systems. It has been investigated in models with constant coefficients [4,31], though it has never been investigated in systems with time-varying parameters.

CRediT authorship contribution statement

J.E. Macías-Díaz: Data curation, Formal analysis, Funding acquisition, Investigation, Project administration, Resources, Software, Supervision, Validation, Writing - original draft, Writing - review & editing. **E. Urenda-Cázares:** Data curation, Formal analysis, Investigation, Writing - original draft. **A. Gallegos:** Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Supervision, Validation, Writing - original draft, Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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